transmission system. The method's feasibility was confirmed with a design example. The actual composite multilayer glass characteristics depend on the accuracy of the thickness and refractive index of the layers, especially for cases where a large number of layers are involved.

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Duality Transformation for Nonreciprocal and Nonsymmetric Transmission Lines

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***Abstract*—Duality transformation is introduced to the theory of generalized (nonreciprocal and nonsymmetric) transmission lines making it possible to find solutions to problems in terms of solutions to dual problems without having to go through the solution process. The generalized transmission lines have emerged when more general media have been introduced to classical waveguide geometries, for example, microstrip lines on chiral substrates. It is seen that there actually exist two duality transformations and the self-dual voltage and current solutions are propagating waves in the transmission line. The transformation can be, e.g., applied to transform a nonsymmetric transmission line to a symmetric one.**

I. Introduction

Duality transformation in electromagnetic theory is based on the symmetry of electric and magnetic quantities in the Maxwell equations. It can be applied to obtain a solution for the dual problem through transforming the solution of the original problem [1]. A careful study for fields and sources in isotropic media showed that there are always two duality transformations of equal importance, [2], [3], and that there exist self-dual quantities invariant to the transformation, which have special physical significance. The theory was later generalized to bi-isotropic media [4] and certain bi-anisotropic media [5].

In circuit theory, the concept of duality has been applied to transform voltages to currents, impedances to admittances, inductances to capacitances, and series circuits to parallel circuits, for example [6], [7]. In transmission-line theory, the line parameters are changed correspondingly in the transformation. It is the purpose of this paper to define the duality transformation to the generalized transmission-line theory introduced recently [8], [9], applicable, e.g., to planar transmission lines on chiral and bi-isotropic substrates [10].

Manuscript received April 9, 1995; revised September 23, 1996.
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Publisher Item Identifier S 0018-9480(97)00276-7.

II. Theory

Let us consider a transmission line with time–harmonic (complex) current $I(x)$ and voltage $U(x)$ functions satisfying the generalized transmission-line equations

$$\frac{d}{dx}\begin{pmatrix}U(x)\\ I(x)\end{pmatrix} + \begin{pmatrix}a & z\\ y & b\end{pmatrix}\begin{pmatrix}U(x)\\ I(x)\end{pmatrix} = \begin{pmatrix}u(x)\\ i(x)\end{pmatrix}. \quad (1)$$

Here we denote the distributed transmission-line circuit parameters by $a,\ z,\ y,\ b$ and the distributed series generator voltage and shunt generator current functions by $u(x)$ and $i(x)$, respectively. The quantities z and y are the distributed series impedance and shunt admittance, respectively. a and b are parameters defining the symmetry and reciprocity of the line [8]. In fact, $a = b$ implies that the impedance parameters of a line section satisfy the symmetry condition $Z_{11} = Z_{22}$. On the other hand, $a = -b$ implies that the impedance parameters satisfy $Z_{12} = Z_{21}$, whence the line is reciprocal [11, p. 158]. The symmetric and reciprocal line with $a = b = 0$ is called the conventional transmission line.

It was also seen [8] that the line is lossless if the parameters satisfy

$$z^* = -z \quad y^* = -y \quad a^* = -b \quad b^* = -a. \quad (2)$$

If we define

$$a = s + d \quad b = s - d \quad s = \frac{a+b}{2} \quad d = \frac{a-b}{2} \quad (3)$$

the quantities $z,\ y$, and s are seen to be imaginary while d is real for a lossless transmission line.

A. The Duality Transformation

The duality transformation is a linear map of the voltage–current pair

$$\begin{pmatrix}U_d\\ I_d\end{pmatrix} = \mathcal{D}\begin{pmatrix}U\\ I\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}U\\ I\end{pmatrix} \quad (4)$$

where $A \cdots D$ are constant (complex) scalar quantities. Operating (1) by the matrix $\mathcal{D}$, we can write the transmission-line equation for the transformed line. The transformation rules for the sources are, then

$$\begin{pmatrix}u_d\\ i_d\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}u\\ i\end{pmatrix} \quad (5)$$

and those for the line parameters

$$\begin{pmatrix}a_d & z_d\\ y_d & b_d\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}a & z\\ y & b\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1}. \quad (6)$$

Let us require that the duality transformation be an involution, i.e., $\mathcal{D}^{-1} = \mathcal{D}$, [3]. This gives a set of conditions for the parameters $A \cdots D$. Ignoring the trivial transformations $\mathcal{D} = \pm\mathcal{I}$, where $\mathcal{I}$ denotes the unit matrix, the conditions are

$$A = -D = \sqrt{1 - BC} \quad (7)$$

implying $\det \mathcal{D} = -1$. Let us introduce the notation

$$A = -D = \sin\theta \quad B = \tau\cos\theta \quad C = \tau^{-1}\cos\theta \quad (8)$$

which takes care of the condition (7) and leaves us two parameters, τ and θ.

Let us now specify the transformation parameters $\tau,\ \theta$ by requiring that a given transmission line (the reference line) with parameters

$a_0,\ z_0,\ y_0,\ b_0$ be transformed to itself. This condition can be written as the matrix equation:

$$\begin{pmatrix}\sin\theta & \tau\cos\theta\\ \tau^{-1}\cos\theta & -\sin\theta\end{pmatrix}\begin{pmatrix}a_0 & z_0\\ y_0 & b_0\end{pmatrix} = \begin{pmatrix}a_0 & z_0\\ y_0 & b_0\end{pmatrix}\begin{pmatrix}\sin\theta & \tau\cos\theta\\ \tau^{-1}\cos\theta & -\sin\theta\end{pmatrix}. \tag{9}$$

The four component equations can be seen to give twice the same two

$$\tau^2 = \frac{z_0}{y_0} \quad \tan\theta = \tau\frac{a_0 - b_0}{2z_0} = \frac{\tau d_0}{z_0} \tag{10}$$

which can be solved as

$$\tau_\pm = \pm\sqrt{\frac{z_0}{y_0}} \quad \tan\theta_\pm = \pm\frac{d_0}{\sqrt{z_0 y_0}}. \tag{11}$$

Thus, there are actually two duality transformations that satisfy the given conditions

$$\mathcal{D}_\pm = \frac{\pm 1}{\sqrt{z_0 y_0 + d_0^2}}\begin{pmatrix}d_0 & z_0\\ y_0 & d_0\end{pmatrix}. \tag{12}$$

The two duality transformations $\mathcal{D}_+$ and $\mathcal{D}_-$ differ only by the sign.

The transformation formulas for the line parameters are the same for both signs and can be written as

$$s_d = s \tag{13}$$

$$\frac{d_d}{\sqrt{z_0 y_0}} = -\frac{d}{\sqrt{z_0 y_0}}\cos 2\theta + \left(\frac{z}{2z_0} + \frac{y}{2y_0}\right)\sin 2\theta \tag{14}$$

$$\frac{z_d}{z_0} = \frac{d}{\sqrt{z_0 y_0}}\sin 2\theta - \frac{z}{z_0}\sin^2\theta + \frac{y}{y_0}\cos^2\theta \tag{15}$$

$$\frac{y_d}{y_0} = \frac{d}{\sqrt{z_0 y_0}}\sin 2\theta + \frac{z}{z_0}\cos^2\theta - \frac{y}{y_0}\sin^2\theta. \tag{16}$$

In terms of the reference-line parameters we have

$$d_d = \frac{d_0(zy_0 + yz_0 + dd_0) - z_0 y_0 d}{z_0 y_0 + d_0^2} \tag{17}$$

$$z_d = \frac{2z_0 dd_0 - zd_0^2 + yz_0^2}{z_0 y_0 + d_0^2} \tag{18}$$

$$y_d = \frac{2y_0 dd_0 - yd_0^2 + zy_0^2}{z_0 y_0 + d_0^2}. \tag{19}$$

It is easy to check that the parameter set $d = d_0,\ z = z_0$ and $y = y_0$ transforms to itself. The parameter $s = (a+b)/2$ transforms to itself in all cases. Thus, the dual of a reciprocal line with $s = 0$ is always reciprocal $s_d = 0$. It is not possible to transform a nonreciprocal line to a reciprocal line.

B. Self-Dual Quantities

By definition, the reference transmission line with the parameters $z_0,\ y_0,\ a_0 = s_0 + d_0$ and $b_0 = s_0 - d_0$ is invariant, or self dual, with respect to the two transformations (12). Let us now find the self-dual voltage and current functions $U_+,\ I_+$ and $U_-,\ I_-$ in the reference line satisfying the two equations:

$$\mathcal{D}_\pm\begin{pmatrix}U_\pm\\ I_\pm\end{pmatrix} = \begin{pmatrix}U_\pm\\ I_\pm\end{pmatrix}. \tag{20}$$

Any voltage and current combination can be written in its self-dual parts as

$$\begin{pmatrix}U\\ I\end{pmatrix} = \begin{pmatrix}U_+\\ I_+\end{pmatrix} + \begin{pmatrix}U_-\\ I_-\end{pmatrix} \tag{21}$$

with

$$\begin{pmatrix}U_\pm\\ I_\pm\end{pmatrix} = \frac{1}{2}(\mathcal{I} + \mathcal{D}_\pm)\begin{pmatrix}U\\ I\end{pmatrix} = \frac{\pm 1}{\gamma_+ - \gamma_-}\begin{pmatrix}\gamma_\pm - b_0 & z_0\\ y_0 & \gamma_\pm - a_0\end{pmatrix}\begin{pmatrix}U\\ I\end{pmatrix} \tag{22}$$

$$\gamma_\pm = s_0 \pm \sqrt{z_0 y_0 + d_0^2} \tag{23}$$

as is readily seen. Outside the sources, the self-dual voltage and current satisfy

$$U_\pm = Z_\pm I_\pm, \quad Z_\pm = \frac{1}{y_0}\left(d_0 \pm \sqrt{z_0 y_0 + d_0^2}\right) = \frac{g_\pm - b_0}{y_0} \tag{24}$$

$$I_\pm = Y_\pm U_\pm, \quad Y_\pm = \frac{1}{z_0}\left(-d_0 \pm \sqrt{z_0 y_0 + d_0^2}\right) = \frac{g_\pm - a_0}{z_0}. \tag{25}$$

Thus, the transmission-line equations for the self-dual quantities outside the sources are

$$\frac{d}{dx}\begin{pmatrix}U_\pm\\ I_\pm\end{pmatrix} = -\begin{pmatrix}a_0 & z_0\\ y_0 & b_0\end{pmatrix}\begin{pmatrix}U_\pm\\ I_\pm\end{pmatrix} = -\begin{pmatrix}a_0 + z_0 Y_\pm & 0\\ 0 & b_0 + y_0 Z_\pm\end{pmatrix}\begin{pmatrix}U_\pm\\ I_\pm\end{pmatrix} = -\gamma_\pm\begin{pmatrix}U_\pm\\ I_\pm\end{pmatrix} \tag{26}$$

where $Z_\pm$ and $Y_\pm$ are the wave impedances and admittances, respectively. The expressions $\gamma_\pm$ defined in (23) represent complex propagation coefficients. The self-dual voltage and current satisfy the same first-order differential equations and their solutions are exponential functions:

$$U_\pm(x) = U_\pm(0)e^{-\gamma_\pm x} \quad I_\pm(x) = I_\pm(0)e^{-\gamma_\pm x}. \tag{27}$$

Thus, the self-dual voltage and current functions are waves propagating in opposite directions on the transmission line.

From the above expressions it is seen that the parameters a_0 and b_0 affect the wave impedances $Z_\pm$ only through their difference, i.e., through the parameter d_0. On the other hand, the propagation coefficients $\gamma_\pm$ depend both on s_0 and d_0. For $d_0 = 0$, the wave impedances satisfy $Z_+ = -Z_-$ and, for $s_0 = 0$, the propagation coefficients satisfy $\gamma_+ = -\gamma_-$, in which case the self-dual waves propagate symmetrically in both directions.

If a section of transmission line has impedance parameters Z_{ij}, an impedance load Z_L is seen through the section as $Z_{\text{in}1}$ from one end and $Z_{\text{in}2}$ from the other end. Their difference can be written as

$$Z_{\text{in}1} - Z_{\text{in}2} = (Z_{11} - Z_{22})\left(1 - \frac{Z_{12}Z_{21}}{(Z_L - Z_{11})(Z_L - Z_{22})}\right). \tag{28}$$

It is seen that this difference vanishes for $Z_{11} = Z_{22}$, which is valid for $d_0 = 0$ [8]. Thus, the line is symmetric even if $\gamma_+ \neq -\gamma_-$, because the wave goes in both directions when reflecting back from the end of the line.

C. The Conventional Line

For a conventional lossless self-dual transmission line with

$$a_0 = b_0 = 0 \quad z_0 = j\omega\ell \quad y_0 = j\omega c \tag{29}$$

where ℓ and c denote real series inductance and shunt capacitance per unit length, the complex propagation coefficients are

$$\gamma_\pm = \pm\sqrt{z_0 y_0} = \pm j\omega\sqrt{\ell c} \tag{30}$$

corresponding to propagation in $\pm x$ directions on the line. The wave impedances become

$$Z_\pm = \pm\sqrt{\frac{z_0}{y_0}} = \pm\sqrt{\frac{\ell}{c}} \tag{31}$$

which is another familiar formula. The duality transformation matrices have the simple appearance in this special case

$$\mathcal{D}_{\pm} = \pm\begin{pmatrix} 0 & Z_0 \\ Y_0 & 0 \end{pmatrix}, \quad Z_0 = Y_0^{-1} = \sqrt{z_0/y_0}. \tag{32}$$

III. Applications

Duality is a basic principle that can be readily applied to transform transmission-line problems to other transmission-line problems. Let us consider two simple examples.

A. Transformation to a Conventional Line

A nonsymmetric transmission line with $d \neq 0$ can always be transformed to a symmetric one with $d_d = 0$. In fact, setting $d_d = 0$ in (17), we obtain a relation for the reference-line parameters $z_0,\ y_0,\ d_0,$

$$d_0(zy_0 + yz_0 + dd_0) - z_0 y_0 d = 0. \tag{33}$$

The parameters (18) and (19) of the transformed line are then

$$z_d = -z + z_0\frac{d}{d_0} \tag{34}$$

$$y_d = -y + y_0\frac{d}{d_0}. \tag{35}$$

We are free to choose the parameters z_0 and y_0, after which d_0 is obtained from (33). For example, the choice $z_0 = z,\ y_0 = y$ gives us

$$d_0 = \frac{zy}{d}\left(\sqrt{1 + d^2/zy} - 1\right) \tag{36}$$

and

$$z_d = z\sqrt{1 + d^2/zy}, \quad y_d = y\sqrt{1 + d^2/zy}. \tag{37}$$

The wave impedances are transformed in simple form

$$Z_{\pm} = \frac{1}{y}\left(d \pm \sqrt{zy + d^2}\right) \to Z_{d\pm} = \pm\sqrt{\frac{z_d}{y_d}} = \pm\sqrt{\frac{z}{y}}. \tag{38}$$

On the other hand, since the parameter s is not transformed to zero, the propagation factors satisfy $\gamma_+ \neq -\gamma_-$ when $s \neq 0$ and $\gamma_+ = -\gamma_-$ when $s = 0$, both before and after the transformation.

B. Impedance-Matching Network

As another example we may consider impedance matching by means of a conventional lossless stub network [11], which transforms the load impedance Z_L to a resistive impedance Z_1. The transformation rule for the impedance is the same for both duality transformations. Assuming that the reference line is reciprocal $(s = 0)$ and lossless (z_0 and y_0 are imaginary and $a_0 = -b_0 = 2d_0$ is real), we can write the impedance transformation rule $Z \to Z_d$ as

$$Z_d = \frac{d_0 Z + z_0}{y_0 Z - d_0}. \tag{39}$$

It is easy to see that the inverse transformation obeys the same rule.

The dual of the matching network is another matching network with impedances changed according to the above rule. In particular, short circuit $Z = 0$ is transformed to the load impedance $-z_0/d_0$, which is imaginary for a lossless reference line (z_0 imaginary and d_0 real) and corresponds to a certain capacitive load. Correspondingly, open circuit $Z = \infty$ is transformed to the impedance d_0/y_0.

Let us require that the dual network matches the same real impedance Z_1 as the original one, i.e., Z_1 is a self-dual impedance. This gives us a condition to the transformation parameters

$$Z_1^2 - 2\kappa Z_0 Z_1 - Z_0^2 = 0 \quad \kappa = \frac{d_0}{\sqrt{z_0 y_0}} \quad Z_0 = \sqrt{z_0/y_0}. \tag{40}$$

We can solve for the real parameter κ

$$\kappa = \frac{Z_1^2 - Z_0^2}{2Z_0 Z_1}. \tag{41}$$

The value Z_0 is still open. If the original load impedance is Z_L, its dual is

$$Z_{Ld} = Z_0\frac{\kappa Z_L + Z_0}{Z_L - \kappa Z_0} = \frac{Z_L Z_1^2 - Z_L Z_0^2 + 2Z_1 Z_0^2}{2Z_1 Z_0 - Z_1^2 + Z_0^2}. \tag{42}$$

By varying the real positive Z_0 we see the range of possible load impedances Z_{Ld}, which can be matched through the dual network. For example, if $Z_0 = Z_1$, we have $Z_{Ld} = Z_1^2/Z_L$.

IV. Conclusion

Duality transformation has been introduced to the theory of transmission lines involving generalized (nonsymmetric and nonreciprocal) transmission lines, recently introduced in the literature. It was seen that there exist two duality transformations that differ by the sign from each other. They can be defined by requiring that a certain transmission line (the reference line) is invariant in the transformations. Self-dual voltage and current functions were seen to be the voltage and current waves on the transmission line. The dual of a nonreciprocal line is always nonreciprocal, but a nonsymmetric line can be transformed to a symmetric line. Since duality is one of the basic properties of transmission lines, it can be applied to transmission-line problems in general. A matching transmission-line circuit is discussed as an example.

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